A simple geometric construction of e

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We consider the exponential curves $y = ba^x$ where a, $b \in R$, $a > 0$, a $\neq 1$, and $b \neq 1$. We show that by fixing one of a or b the points (x_0,y_0) of tangency of $y = ba^x$ and the line passing through the origin (see Figure 1) lie on a straight line. Moreover, the y-ordinates of these points are equal to be.

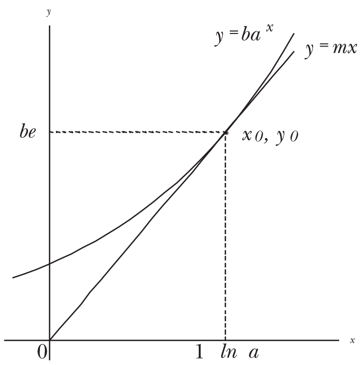

Figure 1

Proposition 1

Let $a \in R$ with $a > 0$, $a \neq 1$ be fixed and let $b \in R$ with $b \neq 0$. Then the coordinates of the points of tangency (x_0,y_0) of the curve $y = ba^x$ and the line $y = mx$ (see Figure 2) are

$$x_0 = \frac{1}{\ln a} \quad \text{and} \quad y_0 = be$$

Proof

The slope of the tangent line to $y = ba^x$ at (x_0,y_0) is

$$m_1 = \frac{d}{dx}\left(ba^x\right)\bigg|_{x=x_0} = b\ln a a^{x_0}$$

The slope of the tangent line through (x_0,y_0) and the origin $(0,0)$ is

$$m_2 = \frac{y_0}{x_0}$$

Equating m_1 with m_2 we obtain :

$$b \cdot \ln a \cdot a^{x_0} = \frac{y_0}{x_0} \Rightarrow b \cdot \ln a \cdot a^{x_0} = \frac{ba^{x_0}}{x_0} \Rightarrow x_0 = \frac{1}{\ln a}$$

and so

$$y_0 = b \cdot a^{x_0} = b \cdot a^{\frac{1}{\ln a}} = b \cdot e$$

Proposition 2

Let $b \in R$ with $b \neq 1$ be fixed and let $a \in R$ with $a > 0$, $a \neq 1$. Then the coordinates of the points of tangency (x_0, y_0) of the curve $y = ba^x$ and the line $y = mx$ (see Figure 3) are

$$x_0 = \frac{1}{\ln a} \quad \text{and} \quad y_0 = be$$

Proof

Similar to Proposition 1. Propositions 1 and 2 suggest the following procedure for constructing geometrically an arbitrary multiple be of e, where $b \neq 0$.

1. Draw the curve $y = ba^x$ where a is an arbitrary real number with $a > 0$, $a \neq 1$.
2. Draw the line tangent to $y = ba^x$ and passing through the origin $(0,0)$.
3. The y-ordinate of the point of tangency is be.

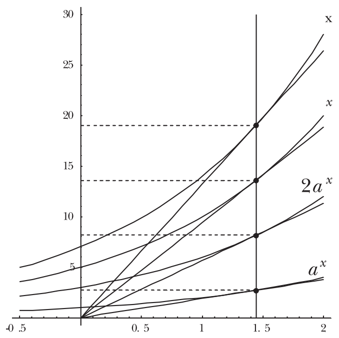

Figure 2

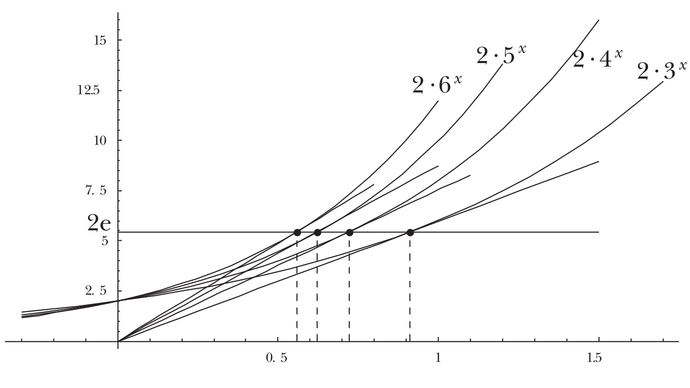

Figure 3